Dynamic energy release rate in couple-stress elasticity

2013 J. Phys.: Conf. Ser. 451 012014

(http://iopscience.iop.org/1742-6596/451/1/012014)

Dynamic energy release rate in couple-stress elasticity

L Morini[1], A Piccolroaz[1] and G Mishuris[2]

[1] Department of Civil, Environmental and Mechanical Engeneering,
University of Trento, Via Mesiano 77, 38123 Trento, Italy
[2]Institute of Mathematical and Physical Sciences,
Aberystwyth University, Ceredigion SY23 3BZ, Wales, U.K.

E-mail: lorenzo.morini@unitn.it

Abstract. This paper is concerned with energy release rate for dynamic steady state crack problems in elastic materials with microstructures. A Mode III semi-infinite crack subject to loading applied on the crack surfaces is considered. The micropolar behaviour of the material is described by the theory of couple-stress elasticity developed by Koiter. A general expression for the dynamic J-integral including both traslational and micro-rotational inertial contributions is derived, and the conservation of this integral on a path surrounding the crack tip is demonstrated.

1. Introduction

The explicit evaluation of the energy release rate and the analysis of its variation in function of the microstructural parameters and of the crack velocity are crucial issues for studying crack propagation stability in couple stress elastic materials [11]. For this reason, a general expression for the dynamic J-integral associated to a semi-infinite Mode III steady-state crack is obtained, and the energy release rate is derived by means of the conservation of this integral. A general procedure for the explicit evaluation of the J-integral, considering a rectangular-shaped contour surrounding the crack tip [3, 4, 5], is also illustrated.

2. Problem formulation

A Cartesian coordinate system $(0, x_1, x_2, x_3)$ centred at the crack-tip at time $t = 0$ is assumed. The micropolar behavior of the material is described by the indeterminate theory of couple stress elasticity [6]. The non-symmetric Cauchy stress tensor $\boldsymbol{t}$ can be decomposed into a symmetric part $\boldsymbol{\sigma}$ and a skew-symmetric part $\boldsymbol{\tau}$, namely $\boldsymbol{t} = \boldsymbol{\sigma} + \boldsymbol{\tau}$. The reduced tractions vector $\boldsymbol{p}$ and couple stress tractions vector $\boldsymbol{q}$ are defined as

$$\boldsymbol{p} = \boldsymbol{t}^T \boldsymbol{n} + \frac{1}{2}\nabla\mu_{nn} \times \boldsymbol{n}, \quad \boldsymbol{q} = \boldsymbol{\mu}^T \boldsymbol{n} - \mu_{nn}\boldsymbol{n}, \tag{1}$$

where $\boldsymbol{\mu}$ is the couple stress tensor, $\boldsymbol{n}$ denotes the outward unit normal and $\mu_{nn} = \boldsymbol{n} \cdot \boldsymbol{\mu}\boldsymbol{n}$. For the dynamic antiplane problem, the conditions of dynamic equilibrium of forces and moments, taking into condideration rotational inertia, and neglecting body forces and body couples, write

$$\sigma_{13,1} + \sigma_{23,2} + \tau_{13,1} + \tau_{23,2} = \rho\ddot{u}_3, \quad \mu_{11,1} + \sigma_{21,2} + 2\tau_{23} = J\ddot{\varphi}_1, \quad \mu_{12,1} + \sigma_{22,2} - 2\tau_{13} = J\ddot{\varphi}_2, \tag{2}$$

where φ_1 and φ_2 are components of the rotations vector, defined as:

$$\varphi_1 = \frac{1}{2}u_{3,2}, \quad \varphi_2 = -\frac{1}{2}u_{3,1}, \tag{3}$$

$\boldsymbol{\rho}$ is the mass density and J is the rotational inertia. The stresses and couple stresses can be expressed in terms of the out-of plane displacement u_3 [11, 12]:

$$\sigma_{13} = Gu_{3,1}, \quad \sigma_{23} = Gu_{3,2}, \tag{4}$$

$$\tau_{13} = -\frac{G\ell^2}{2}\Delta u_{3,1} + \frac{J}{4}\ddot{u}_{3,1}, \quad \tau_{23} = -\frac{G\ell^2}{2}\Delta u_{3,2} + \frac{J}{4}\ddot{u}_{3,2}, \tag{5}$$

$$\mu_{11} = -\mu_{22} = G\ell^2(1+\eta)u_{3,12}, \quad \mu_{21} = G\ell^2(u_{3,22} - \eta u_{3,11}), \quad \mu_{12} = -G\ell^2(u_{3,11} - \eta u_{3,22}). \tag{6}$$

where Δ denotes the Laplace operator, J is the rotational inertia, G is the elastic shear modulus, ℓ and η the couple stress parameters introduced by Koiter [6], with $-1 < \eta < 1$. Both material parameters ℓ and η depend on the microstructure and can be connected to the material characteristic lengths in bending and in torsion [12], namely $\ell_b = \ell/\sqrt{2}$ and $\ell_t = \ell\sqrt{1+\eta}$. Typical values of ℓ_b and ℓ_t for some classes of materials with microstructure can be found in references [7] and [8].
Substituting expressions (4), (5) and (6) in the dynamic equilibrium equation $(2)_{(1)}$, the following equation of motion is derived [11]:

$$G\Delta u_3 - \frac{G\ell^2}{2}\Delta\Delta u_3 + \frac{J}{4}\Delta\ddot{u}_3 = \rho\ddot{u}_3. \tag{7}$$

We assume that the crack propagates with a constant velocity v straight along the x_1-axis and is subjected to reduced force traction p_3 applied on the crack faces, moving with the same velocity v, whereas reduced couple traction q_1 is assumed to be zero,

$$p_3(x_1, 0^\pm, t) = \mp\tau(x_1 - vt), \quad q_1(x_1, 0^\pm, t) = 0, \quad \text{for} \quad x_1 - vt < 0. \tag{8}$$

We also assume that the function τ decays at infinity sufficiently fast and it is bounded at the crack tip. These requirements are the same requirements for tractions as in the classical theory of elasticity.
It is convenient to introduce a moving framework $x = x_1 - vt$, $y = x_2, z = x_3$. By assuming that the out of plane displacement field has the form $u_3(x_1, x_2, t) = w(x, y)$, then the equation of motion (7) writes:

$$\Delta w - \frac{\ell^2}{2}\Delta\Delta w = m^2\left(w_{,xx} - h_0^2\ell^2\Delta w_{,xx}\right) \tag{9}$$

where $m = v/c_s$ is the crack velocity normalized to the shear waves speed c_s, and $h_0 = \sqrt{J/4\rho}/\ell$ is the normalized rotational inertia defined in Mishuris et al., 2012.
According to (1), the non-vanishing components of the reduced force traction and reduced couple traction vectors along the crack line $y = 0$, where $\boldsymbol{n} = (0, \pm 1, 0)$, can be written as

$$p_3 = t_{23} + \frac{1}{2}\mu_{22,x}, \quad q_1 = \mu_{21}, \tag{10}$$

respectively. By using $(4)_2$, $(6)_{1,2}$, $(5)_2$, and (10), the loading conditions (8) on the upper crack surface require the following conditions for the function w:

$$w_{,y} - \frac{\ell^2}{2}\left[(2+\eta-2m^2h_0^2)w_{,xx} + w_{,yy}\right]_{,y} = -\frac{1}{G}\tau(x), \quad w_{,yy} - \eta w_{,xx} = 0, \quad \text{for} \quad x < 0, \quad y = 0^+. \tag{11}$$

Ahead of the crack tip, the skew-symmetry of the Mode III crack problem requires

$$w = 0, \quad w_{,yy} - \eta w_{,xx} = 0, \quad \text{for} \quad x > 0, \quad y = 0^+. \tag{12}$$

Note that the ratio η enters the boundary conditions (11)-(12), but it does not appear into the governing PDE (9).

3. Dynamic energy release rate

For couple stress elastic materials, the energy flux for a dynamic crack propagating along x_1-axis is given by:

$$F(\Gamma) = \int_\Gamma \left[(W+T) v n_1 + \mathbf{t}^T \mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\mu}^T \mathbf{n} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} \right] ds, \tag{13}$$

where $\mathbf{n}$ is an outward unit normal on Γ. The strain-energy density W and the kinetic enrgy density T are given by

$$W = \frac{1}{2} \left(\boldsymbol{\sigma} \cdot \nabla \mathbf{u} + \boldsymbol{\mu}^T \cdot \nabla \boldsymbol{\varphi} \right), \quad T = \frac{1}{2} \left(\rho \, |\dot{\mathbf{u}}|^2 + J \, |\dot{\boldsymbol{\varphi}}|^2 \right). \tag{14}$$

Considering a Mode III steady-state crack prapagating at constant velocity v along the x_1-axis, and using the expression for the the out-of-plane defined in Section 2, in the moving framework (x, y, z) the energy flux (13) assumes the special form

$$F(\Gamma) = v \int_\Gamma \left[(W+T) n_x - \mathbf{t}^T \mathbf{n} \cdot \mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T \mathbf{n} \cdot \boldsymbol{\varphi}_{,x} \right] ds, \tag{15}$$

then the generalized J-integral for an antiplane dynamic steady-state crack in couple-stress elastic materials can be defined:

$$\begin{aligned} \mathcal{J} = \frac{F(\Gamma)}{v} &= \int_\Gamma \left[(W+T) n_x - \mathbf{t}^T \mathbf{n} \cdot \mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T \mathbf{n} \cdot \boldsymbol{\varphi}_{,x} \right] ds = \\ &= \int_\Gamma \left[(W+T) \mathbf{n} - \mathbf{p} \cdot \mathbf{e}_z \nabla w - (\nabla \boldsymbol{\varphi})^T \mathbf{q} \right] \cdot \mathbf{e}_x ds \end{aligned} \tag{16}$$

The J-integral (16) has been proved to be path independent, the details of the demonstration are reported in the next Section, and it is the generalization of the static expressions derived by Freund and Hutchinson [2] and Lubarda and Markenscoff [9] to the antiplane dynamic steady state case. The dynamic energy release rate is then defined by the limit [1]:

$$\mathcal{G} = \lim_{\Gamma \to 0} \int_\Gamma \left[(W+T) n_x - \mathbf{t}^T \mathbf{n} \cdot \mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T \mathbf{n} \cdot \boldsymbol{\varphi}_{,x} \right] ds. \tag{17}$$

4. Conservation of the J-integral

In this Section, we demonstrate that the dynamic J-integral expression (16) is path independent. Considering a closed oriented path formed by two crack tip contours Γ_1 and Γ_2 and by the segments of the crack faces of length d that connect the ends of these contours and using the notation introduced in Section 2, the energy flux integral corresponding to this entire closed path Γ_{tot} for a Mode III steady state crack in couple-stress materials is given by

$$F(\Gamma_{tot}) = F(\Gamma_2) - F(\Gamma_1) = v \oint_{\Gamma_{tot}} \left[(W+T) n_x - \mathbf{t}^T \mathbf{n} \cdot \mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T \mathbf{n} \cdot \boldsymbol{\varphi}_{,x} \right] ds, \tag{18}$$

then from the definition (16) we derive

$$\mathcal{J}(\Gamma_{tot}) = \mathcal{J}(\Gamma_2) - \mathcal{J}(\Gamma_1) = \oint_{\Gamma_{tot}} \left[(W+T)n_x - \mathbf{t}^T\mathbf{n}\cdot\mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T\mathbf{n}\cdot\boldsymbol{\varphi}_{,x}\right] ds, \tag{19}$$

where the notation $\mathcal{J}(\Gamma_1)$ and $\mathcal{J}(\Gamma_2)$ denotes that the dynamic J-integral (16) is evaluated respect to the crack tip contours Γ_1 and Γ_2, respectively. Applying the divergence theorem to the (19) we obtain and remembering that $n_x = \mathbf{n}\cdot\mathbf{e}_x$, we obtain

$$\mathcal{J}\Gamma_2) - \mathcal{J}(\Gamma_1) = \int_{A_{tot}} \nabla\cdot\left[(W+T)\mathbf{e}_x - \mathbf{t}^T\mathbf{e}_z w_{,x} - \boldsymbol{\mu}^T\boldsymbol{\varphi}_{,x}\right] dA, \tag{20}$$

where A_{tot} is the area within the closed area. For the antiplane steady-state problem the strain elastic energy density and the kinetic energy density are given by

$$W = \frac{1}{2}(\sigma_{13}w_{,x} + \sigma_{23}w_{,y} + \mu_{11}\varphi_{1,x} + \mu_{12}\varphi_{2,x} + \mu_{21}\varphi_{1,y} + \mu_{22}\varphi_{2,y}), \tag{21}$$

$$T = \frac{v^2}{2}(\rho w_{,x}^2 + J\varphi_{1,x}^2 + J\varphi_{2,x}^2), \tag{22}$$

the first term of the integral (20) is the given by

$$\begin{aligned} \nabla\cdot[(W+T)\mathbf{e}_x] &= (W+T)_{,x} = v^2\left(\rho w_{,xx}w_{,x} + J\varphi_{1,xx}\varphi_{1,x} + J\varphi_{2,xx}\varphi_{2,x}\right) + \\ &+ \frac{1}{2}\left(\sigma_{13,x}w_{,x} + \sigma_{13}w_{,xx} + \sigma_{23,x}w_{,y} + \sigma_{23}w_{3,yx}\right) + \\ &+ \frac{1}{2}\left(\mu_{11,x}\varphi_{1,x} + \mu_{11}\varphi_{1,xx} + \mu_{12,x}\varphi_{2,x} + \mu_{12}\varphi_{2,xx} + \right. \\ &+ \left. \mu_{21,x}\varphi_{1,y} + \mu_{21}\varphi_{1,yx} + \mu_{22,x}\varphi_{2,y} + \mu_{22}\varphi_{2,yx}\right). \end{aligned} \tag{23}$$

Taking into account the dynamic equilibrium conditions (2), the second term can be written as follows

$$\begin{aligned} \nabla\cdot(\mathbf{t}^T\mathbf{e}_z w_{,x}) &= (\nabla\cdot\mathbf{t}^T)\cdot\mathbf{e}_z w_{,x} + \mathbf{t}^T\cdot\nabla w_{,x} = \\ &= \rho\ddot{u}w_{,x} + (\sigma_{13}+\tau_{13})\,w_{,xx} + (\sigma_{23}+\tau_{23})\,w_{,yx} = \\ &= \rho v^2 w_{,xx}w_{,x} + (\sigma_{13}+\tau_{13})\,w_{,xx} + (\sigma_{23}+\tau_{23})\,w_{,yx}, \end{aligned} \tag{24}$$

while the third

$$\begin{aligned} \nabla\cdot(\boldsymbol{\mu}^T\boldsymbol{\varphi}_{,x}) &= (\nabla\cdot\boldsymbol{\mu}^T)\cdot\boldsymbol{\varphi}_{,x} + \boldsymbol{\mu}^T\cdot\nabla\boldsymbol{\varphi}_{,x} = \\ &= (J\ddot{\varphi}_1 - 2\tau_{23})\,\varphi_{1,x} + (J\ddot{\varphi}_2 + 2\tau_{13})\,\varphi_{2,x} + \mu_{11}\varphi_{1,x} + \mu_{12}\varphi_{2,x} + \mu_{21}\varphi_{1,y} + \mu_{22}\varphi_{2,y} = \\ &= \left(Jv^2\varphi_{1,xx} - 2\tau_{23}\right)\varphi_{1,x} + \left(Jv^2\varphi_{2,xx} + 2\tau_{13}\right)\varphi_{2,x} + \mu_{11}\varphi_{1,x} + \mu_{12}\varphi_{2,x} + \mu_{21}\varphi_{1,y} + \mu_{22}\varphi_{2,y}. \end{aligned} \tag{25}$$

Substituting (23), (25) and (25) into the integral (20) and expressing φ_1 and φ_2 in function of the displacement by means of relations (3), we obtain

$$\begin{aligned} \mathcal{J}(\Gamma_2) - \mathcal{J}(\Gamma_1) &= \int_{A_{tot}} \left[\frac{1}{2}\left(\sigma_{13,x}w_{,x} + \sigma_{23,x}w_{,y} + \frac{1}{2}\left(\mu_{11,x}w_{,yx} - \mu_{12,x}w_{,xx} + \mu_{21,x}w_{,yy} - \mu_{22,x}w_{,xy}\right)\right) - \right. \\ &- \left. \frac{1}{2}\left(\sigma_{13}w_{,xx} + \sigma_{23}w_{,yx} + \frac{1}{2}\left(\mu_{11}w_{,yxx} - \mu_{12}w_{,xxx} + \mu_{21}w_{,yyx} - \mu_{22}w_{,xyx}\right)\right)\right] dA, \end{aligned} \tag{26}$$

finally, introducing into expressions (4) and (6), which define stress and couple-stress tensors in function of the derivatives of the displacement, we get:

$$\begin{aligned}\mathcal{J}(\Gamma_2)-\mathcal{J}(\Gamma_1) &= \frac{G\ell^2}{2}\int_{A_{tot}}\left[\left(w_{,xx}w_{,xxx}+w_{,yy}w_{,yyx}-\eta\left(w_{,yyx}w_{,xx}+w_{,xxx}w_{,yy}\right)\right)-\right.\\ &-\left.\left(w_{,xx}w_{,xxx}+w_{,yy}w_{,yyx}-\eta\left(w_{,yyx}w_{,xx}+w_{,xxx}w_{,yy}\right)\right)\right]dA=0. \qquad (27)\end{aligned}$$

We have demonstrated that the J-integral (16) is path independent, as a consequence the dynamic energy release rate (17) can be evaluated by choosing an arbitrary path Γ surrounding the crack tip and making the limit $\Gamma \to 0$.

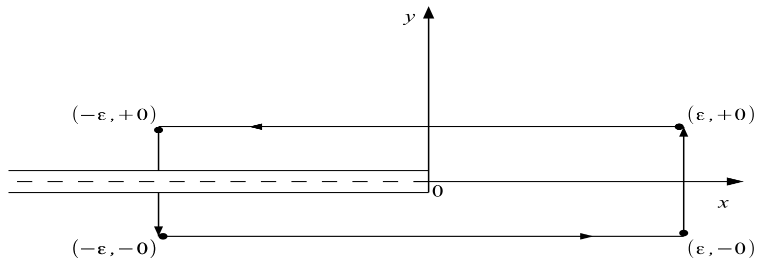

Figure 1. Rectangular-shaped contour around the crack-tip

5. Evaluation of the energy release rate

For the evaluation of the energy release rate (17), a rectangular-shaped contour contour Γ with vanishing height along the y-direction and with $\varepsilon \to +0$ reported in Fig.1. Such contour was first introduced by Freund [1] and recently used by Gourgiotis and Georgiadis [3, 4, 5] and it permits using solely the asymptotic near-tip stress and displacement fields.

Considering the moving framework in Fig.1 with the origin at the crack tip, the cartesian components of the outward unit vector normal to Γ are $\mathbf{n} = (n_x, n_y, 0)$, and the generalized J-integral (16) becomes:

$$\begin{aligned}\mathcal{J} &= \int_\Gamma \left[(W+T)n_x-(t_{13}n_x+t_{23}n_y)w_{,x}-(\mu_{11}n_x+\mu_{21}n_y)\varphi_{1,x}-(\mu_{12}n_x+\mu_{22}n_y)\varphi_{2,x}\right]ds=\\ &=\int_\Gamma\left[(W+T)-t_{13}w_{,x}-(\mu_{11}\varphi_{1,x}+\mu_{12}\varphi_{2,x})\right]dy-\int_\Gamma\left[t_{23}w_{,x}+(\mu_{21}\varphi_{1,x}+\mu_{22}\varphi_{2,x})\right]dx, \qquad (28)\end{aligned}$$

In order to avaluate the energy release rate, we allow the height of the rectangular path reported in Fig.1 to vanish. In this limit, the first integral of the (28) is zero. It is also important to note that anti-symmetry conditions (11) together with boundary conditions (12) provide that the reduced traction $q_1 = \mu_{21}$ is zero along the whole crack line $y = 0$, where $\mathbf{n} = (0, \pm 1, 0)$. Consequently, the energy release rate (17) becomes:

$$\begin{aligned}\mathcal{G} &= -2\lim_{\varepsilon\to+0}\left\{\int_{-\varepsilon}^{+\varepsilon}\left[t_{23}(x,0^+)w_{,x}(x,0^+)+\mu_{22}(x,0^+)\varphi_{2,x}(x,0^+)\right]dx\right\}\\ &= -2\lim_{\varepsilon\to+0}\left\{\int_{-\varepsilon}^{+\varepsilon}\left[t_{23}(x,0^+)w_{,x}(x,0^+)-\frac{1}{2}\mu_{22}(x,0^+)w_{,xx}(x,0^+)\right]dx\right\}. \qquad (29)\end{aligned}$$

Asymptotic expressions for the total shear stress t_{23}, the couple stress field μ_{22}, and the out-of-plane diplacement w have been derived by Radi [12] and Piccoloraz et al. [10]. These fields exhibit the following behavior near to the crack tip:

$$\begin{aligned} t_{23}(x,0^+) &= Af(h_0,\eta,m)x^{-3/2}, \quad x>0, && (30)\\ \mu_{22}(x,0^+) &= Ag(h_0,\eta,m)x^{-1/2}, \ x>0, && (31)\\ w(x,0^+) &= A(-x)^{3/2}, \quad x<0. && (32)\end{aligned}$$

where A is a constant determined by the boundary conditions (12) and depending by the characteristic of the loading applied at the faces $\tau(x)$, and $f(h_0,\eta,m)$ and $g(h_0,\eta,m)$ are factors relating the asymptotics leading term of the displacement to the leading terms of t_{23} and μ_{22}, respectively. Now, by substituting the (30) and the (32) in the general formula (29), we finally derive:

$$\mathcal{G} = -2\lim_{\varepsilon\to+0}\left\{A\left[f(h_0,\eta,m)\int_{-\varepsilon}^{+\varepsilon} x_-^{1/2}x_+^{-3/2}dx - g(h_0,\eta,m)\int_{-\varepsilon}^{+\varepsilon} x_-^{-1/2}x_+^{-1/2}.dx\right]\right\} \tag{33}$$

The two products of distributions $x_-^{1/2}, x_-^{-1/2}$ and $x_+^{-3/2}, x_+^{-1/2}$ are obtained through the use of Fisher's theorem, that's leads to the operational relation [3]:

$$(x_-)^{\gamma}(x_+)^{-1-\gamma} = -\frac{\pi\delta(x)}{2\sin(\pi\gamma)}, \text{ with } \gamma \neq -1,-2,-3\ldots, \tag{34}$$

where $\delta(x)$ is the Dirac delta distribution. Then, applying the relation (34) to the (33) and considering the fundamental property of the Dirac delta distribution $\int_{-\varepsilon}^{+\varepsilon}\delta(x)dx = 1$, we finally get:

$$\mathcal{G} = A\pi\left(f(h_0,\eta,m) + g(h_0,\eta,m)\right). \tag{35}$$

6. Conclusions

A general expression for the dynamic J-integral valid for antiplane crack in a couple stress elastic material has been derived. A direct procedure for the evaluation of the energy release rate, based on the assumption of a rectangular-shaped path around the crack tip and requiring only asymptotics expressions of stresses and displacement, has been illustrated. The proposed method can be used for evaluating the energy release rate for many dynamic Mode III problems in couple-stress elastic materials characterized by different loading configurations acting on the crack faces.

7. Acknowledgements

L. M. and A. P. gratefully thank financial support of the Italian Ministry of Education, University and Research in the framework of the FIRB project 2010 "Structural mechanics models for renewable energy applications", G. M. gratefully acknowledges the support from the European Union Seventh Framework Programme under contract number PIAP-GA-2011-286110-INTERCER2.

References

[1] Freund LB 1990, *Dynamic Fracture Mechanics,* (Cambridge University Press, Cambridge, UK).

[2] Freund LB and Hutchinson JW 1985, "High strain-rate crack growth in rate-dependent plastic solids", *J. Mech. Phys. Solids* **33**: 169-91.

[3] Georgiadis HG, "The mode III crack problem in microstructured solids governed by dipolar gradient elasticity: static and dynamic analysis", *J. Appl. Mech.*,**70**: 517-30.

[4] Gourgiotis PA and Georgiadis HG 2007, "Distributed dislocation approach for cracks in couple-stress elasticity: shear modes", *Int. J. Fract.* **147**: 83-102.
[5] Gourgiotis PA and Georgiadis HG 2008, "An approach based on distributed dislocations and disclinations for crack problems in couple-stress elasticity", *Int. J. Solids Struct.* **45**: 5521-39.
[6] Koiter WT 1964 "Couple-stresses in the theory of elasticity, I and II", *Proc. Kon. Nederl. Akad. Wetensch (B)* **67**: 17-44.
[7] Lakes RS 1986, "Experimental microelasticity of two porous solids", *Int. J. Solids Struct.*, **22**: 55-63.
[8] Lakes RS 1995, "Experimental methods for study of Cosserat elastic solids and other generalized elastic continua", in *Continuum Models for Materials with Micro-structure* , (John Wiley, New York), pp. 1-22.
[9] Lubarda VA and Markenscoff X 2000, "Conservation integrals in couple stress elasticity", *J. Mech. Phys. Solids* **48**: 553-64.
[10] Piccolroaz A, Mishuris G and Radi E 2012, "Mode III interfacial crack in the presence of couple stress elastic materials", *Eng. Fract. Mech.* **80**: 60-71.
[11] Mishuris G, Piccolroaz A and Radi E 2012, "Steady-state propagation of a Mode III crack in couple stress elastic materials" *Int. J. Eng. Sci.* **61**: 112-28.
[12] Radi E 2008, "On the effects of characteristic lengths in bending and torsion on Mode III crack in couple stress elasticity", *Int. J. Solids Struct.* **45**: 3033-58.